$$\bar{\sigma}_2 = \frac{1}{8(3)^{1/2}}\left(\frac{h}{a}\right)\left(\frac{1}{s}\right)\left[\frac{-87}{(1+s^2)} - \frac{40}{(1+s^2)^2} + \frac{64}{(1+s^2)^3}\left(\frac{341}{256} + \frac{13s^2}{8}\right) + \frac{2b(-4+b^{-2})}{(1+s^2)^{3/2}} \frac{16b}{(1+s^2)^{5/2}}\right] \quad (7d)$$

where

$$\sigma_T = (3)^{1/2}E\alpha^*T_0/(1-\nu) \quad (7e)$$

Inverting Eqs. (7), the bending stress at the support is

$$\sigma_x^*/\sigma_T = \pm(\sigma_0 + \epsilon\sigma_1 + \epsilon^2\sigma_2 + \ldots),\ \text{at}\ \eta = 0 \quad (8a)$$

$$\sigma_0 = -\int_0^\tau J_0(\lambda)d\lambda \quad (8b)$$

$$\sigma_1 = -\frac{(\pi)^{1/2}}{2(6)^{1/4}\,\Gamma(\frac{3}{4})}\left(\frac{h}{a}\right)^{1/2}\left[\int_0^\tau \lambda^{1/4}J_{1/4}(\lambda)d\lambda - 5\int_0^\tau \lambda^{5/4}J_{5/4}(\lambda)d\lambda - 10\int_0^\tau [\cos(\tau-\lambda)]\lambda^{1/4}J_{1/4}(\lambda)d\lambda\right] \quad (8c)$$

$$\sigma_2 = \frac{1}{256(3)^{1/2}}\left(\frac{h}{a}\right)\left[1336(\cos\tau - 1) - 649\tau\sin\tau - 75\tau^2\cos\tau + 64b(-4+b^{-2})\int_0^\tau \lambda J_1(\lambda)d\lambda - 174b\int_0^\tau \lambda^2(\lambda)d\lambda\right] \quad (8d)$$

where $\Gamma(\frac{3}{4})$ denotes the gamma function with argument $\frac{3}{4}$ and $J_n(\lambda)$ indicates a Bessel function of the first kind of order n. A tabulation of the integral appearing in Eq. (8b) is given in Ref. 3.

When the shell is subjected to a Heaviside pressure loading with amplitude P_o, the analogous stresses follow from an integration of the solutions for the impulsively loaded cone of Ref. 1. They are

$$\sigma_0 = \int_0^\tau J_0(\lambda)d\lambda \quad (9a)$$

$$\sigma_1 = \frac{(\pi)^{1/2}}{2(6)^{1/4}\,\Gamma(\frac{3}{4})}\left(\frac{h}{a}\right)^{1/2}\left[\int_0^\tau \lambda^{1/4}J_{1/4}(\lambda)d\lambda + (\tfrac{5}{3}) \times \int_0^\tau \lambda^{5/4}J_{5/4}(\lambda)d\lambda\right] \quad (9b)$$

$$\sigma_2 = \frac{1}{256(3)^{1/2}}\left(\frac{h}{a}\right)\left[(-280 + 75\tau^2)\cos\tau + 280 - 231\tau\sin\tau\right] + (\tfrac{1}{2})\int_0^\tau \lambda J_1(\lambda)d\lambda \quad (9c)$$

where the stress corresponding to σ_T is

$$\sigma_P = (3)^{1/2}(a/h)P_0 \quad (9d)$$

Results

Plots of the dimensionless bending stress on the outer shell surface at the clamped support vs dimensionless time with $\alpha = 10°$ and $a/h = 10$ are shown in Figs. 2 and 3 for the Heaviside heat addition and pressure loading, respectively. As indicated in Figs. 2 and 3, the first-order correction furnishes a significant contribution to the first peak value of stress. Further correction terms appear to be superfluous for the time interval of interest since the second-order correction is negligible during this time period. As in Ref. 1, the validity of the solutions for σ_1 and σ_2 decreases with increasing time due to the presence of secular terms in these equations.

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Use of Hot Wires in Low-Density Flows

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Nomenclature

A	= cross sectional area of hot wire
a	= defined Eq. (10)
C	= constant (function of $\bar{\alpha}$, d and l)
$F(S), f(S), g(S)$	= functions of molecular speed ratio S
I	= current
K_w	= thermal conductivity of wire at temperature T_w
k	= thermal conductivity of air
l	= length of wire
M	= Mach Number
Pr	= Prandtl Number
R	= Wire resistance
$\bar{R}$	= $(T_a - T_s)/(T_i - T_s)$
Re	= Reynolds Number
S	= speed ratio $(\gamma/2)^{1/2}M$
s	= defined Eq. (10)
T	= temperature
T_i	= temperature of wire at which there is no change in current in both the vacuum and flow
u	= velocity
α	= temperature-resistivity coefficient of wire, i.e. $\alpha = (R - R_r)/R_r(T - T_r)$
$\bar{\alpha}$	= accommodation coefficient
μ	= viscosity
ξ	= endlossparameter $(d/l)[(K_w/k_0)(1/Nu_0)]^{1/2}{}_{I\to 0}$
ν	= s/a
σ	= defined Eq. (3)
σ_1	= defined Eq. (4)
ρ	= density
ψ_N	= Nusselt Number correction factor

Subscripts

$(\ )_a$	= adiabatic
$(\ )_{corr}$	= corrected value
$(\ )_d$	= based on wire diameter
$(\ )_{fs}$	= freestream value
$(\ )_m$	= measured
$(\ )_0$	= evaluated at stagnation temperature
$(\ )_r$	= evaluated at reference temperature
$(\ )_s$	= hot wire support
$(\ )_{vac}$	= vacuum
$(\ )_w$	= evaluated at wire temperature
$(\ )_\infty$	= length-average of infinitely long wire

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Introduction

In order to reduce hot wire measurements to flow parameters, it is necessary to accurately determine the end loss correction factors. The end loss effect arises because part of the heat is conducted down the support needles, and this end loss is a function of the wire adiabatic temperature and support temperature. In low-density flows, it is necessary to use extremely long wires to minimize (or eliminate) the end losses, but due to frequent breakage it is more desirable to use moderate length wires. A method is presented in this note whereby the hot wire end loss corrections can be directly and accurately determined. From a calibration of the wire in the freestream and in vacuum, a calibration constant for a particular wire can be determined. The hot wires have been used to study the near wake of cylinders in a Mach 6, low-density flow.

Analysis

It can be proved that the length-average wire temperature T_w is given by Refs. 1–3,

$$T_w = T_{w,\infty} - (T_{w,\infty} - T_s)\beta \tanh 1/\beta \quad (1)$$

where

$$T_{w,\infty} = \sigma_1/\sigma, \quad \beta = 2/\sigma^{1/2}l \quad (2)$$

$$\sigma = [\pi k_0 Nu_0 - (I^2R_r/l)\alpha]/K_wA_w \quad (3)$$

and

$$\sigma_1 = [\pi k_0 Nu_0 T_a + (I^2R_r/l)(1 - \alpha T_r)]/K_wA_w \quad (4)$$

Equation (1) is obtained by making the following assumptions[1,2]: a) the length-average heat transfer coefficient is a constant, b) the temperature of each support is T_s, and c) the wire thermal conductivity K_w is constant. For zero overheat($I \rightarrow 0$), Eq. (1) becomes

$$T_{am} = T_a - (T_a - T_s)\xi \tanh 1/\xi \quad (5)$$

where

$$\xi = \lim_{I\rightarrow 0}\beta$$

and T_a is the adiabatic temperature for an infinitely long wire (i.e. $l/d \rightarrow \infty$)

Hence

$$\xi = (d/l)[(K_w/k_0)1/Nu_0]^{1/2}{}_{I\rightarrow 0} \quad (6)$$

Similarly, the Nusselt Number for a finite length wire at zero overheat can be related to the infinite length value by the equation[3]

$$\psi_N = Nu_0/Nu_m = (1 - \xi \tanh 1/\xi)R_a/R_{am} \quad (7)$$

These correction formulas and those of Refs. 1 and 4 entail a knowledge of the physical characteristics of the wire d, l and K_w. Once these quantities are known, the measured values of adiabatic temperature and Nusselt Number can be corrected using an iterative scheme to give the values for an infinitely long wire. In practice, however, it is often difficult to measure the diameter and "effective" length of the wire. An error in either of these measurements induces a corresponding error in the correction factors. The following analysis calculates the correction factors directly. A quantity $\bar{R} = (T_a - T_s)/(T_i - T_s)$ is introduced. This quantity $\bar{R}$ was first proposed by Dewey.[6] T_i is the temperature of the wire at which there is no change in current in both vacuum and flow. This temperature is obtained in practice by plotting the wire resistance as a function of I^2 or I^2R_w in both vacuum and the flow and noting the intersection of the two curves. Equations (3) and (4) can be written as

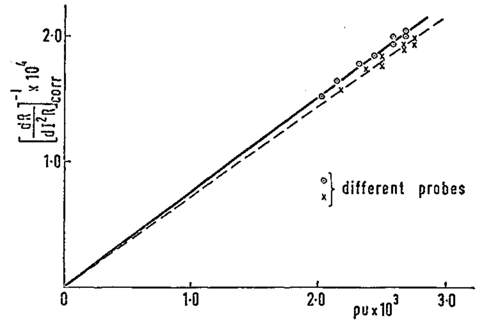

Fig. 2 Typical hot wire calibrations.

$$\sigma l^2/4 = a^2(\nu^2 - 1) \quad (8)$$

$$\sigma_1 l^2/4 = a^2[\nu^2 T_a - (\alpha T_r - 1)/\alpha] \quad (9)$$

where

$$\nu = s/a, \quad s^2 = (l/d)^2(k_0/K_w)Nu_0$$

and

$$a^2 = I^2R_r\alpha l/\pi d^2 K_w \quad (10)$$

Equation (2) becomes

$$T_{w,\infty} = \{\nu^2 T_a - [(\alpha T_r - 1)/\alpha]\}/(\nu^2 - 1) \quad (11)$$

and Eq. (1) becomes

$$T_w = \frac{\nu^2 T_a - (\alpha T_r - 1)/\alpha}{\nu^2 - 1} - \left[\frac{\nu^2 T_a - (\alpha T_r - 1)/\alpha}{\nu^2 - 1} - T_s\right]\frac{\tanh ab}{ab} \quad (12)$$

where $b^2 = \nu^2 - 1$.

In vacuum, the convection term is zero, i.e., $s = 0(\nu = 0)$ Therefore

$$T_{w,\text{vac}} = [(\alpha T_r - 1)/\alpha](1 - \tan a/a) + T_{s,\text{vac}}\tan a/a \quad (13)$$

Hence when $T_w = T_{w,\text{vac}} = T_i$ it can be proved that

$$\nu^2\bar{R} = \frac{\nu^2 - 1}{1 - \tanh ab/ab} + \frac{1}{1 - (\tan a/a) - \beta} \quad (14)$$

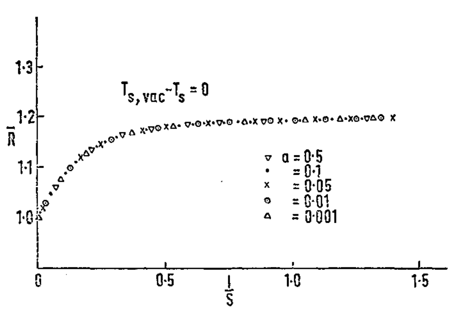

Fig. 1 Hot wire parameter $\bar{R}$.

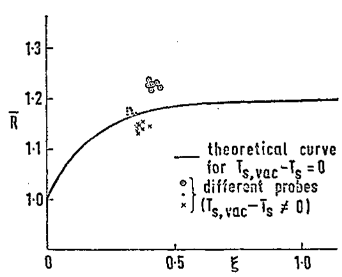

Fig. 3 Variation of calibrated $\bar{R}$ with end loss parameter ξ.

where

$$\beta = [\alpha(T_{s,\text{vac}} - T_s)\tan a/a]/[\alpha(T_s - T_r) + 1]$$

Calibration and Discussion

Figure 1 shows the variation of $\bar{R}$ as a function of $1/s$ for various values of the parameter a for $(T_{s,\text{vac}} - T_s) = 0$. It can be seen that for a fairly wide range of values of a, the parameter $\bar{R}$ is independent of a. Furthermore, it can be shown that for small values of a, $\bar{R}$ approaches the limit 1.2 as $1/s$ increases.

The effect of overheat, $\tau = (T_w - T_a)/T_a$, on Nusselt Number for $\tau < 0.4$ is found to be small,[1,4] and therefore $1/s = \xi$. For most experiments in low density, unheated flows, the Nusselt Number correction factor $\psi_N > 0.6$ and $\xi = 1/s > 0.4$. From Fig. 1 for $1/s > 0.4$, $\bar{R}$ is virtually a constant. In practice, this value of $\bar{R}$ is obtained from a freestream calibration. Hence the measured values of adiabatic temperature and Nusselt Number can be corrected directly to give the values for an infinitely long wire, without having to measure the diameter and effective length of the wire, and without having to use an iteration process.

The freestream calibration was performed in a Mach 6, unheated low-density airstream.[3] The hot wires were platinum-plated tungsten of nominal diameter 0.0002 in. and the distance between the support needles was approximately $\frac{3}{8}$ inch. Thermocouples were soldered to within 0.005 in. of the tip of one of the support needles, so as to measure T_s directly. For any given wire, Nusselt Number is proportional to $(1/k_0)(dR/dI^2R)^{-1}$. Furthermore, for free molecule flow, T_a and Nu_0 are given by the following relationships[7]:

$$Nu_0 = [(\gamma - 1)/2\pi^{3/2}]\bar{\alpha}Pr_0Re_{0,d}g(S)/S \tag{15}$$

and

$$T_a/T_0 = \{1 + [(\gamma - 1)/2]M^2\}^{-1}f(S)/g(S) = F(S) \tag{16}$$

where $f(S)$ and $g(S)$ depend only on the speed ratio S and the number of excited degrees of freedom. These functions are tabulated in Ref. 7 for both monatomic and diatomic perfect gases. The functions $g(S)/S$ and $F(S)$ approach constants for $S > 2.5$. The limiting values are

$$\left[\frac{g(S)}{S}\right]_{S\to\infty} = \frac{\gamma + 1}{\gamma - 1}\pi^{3/2} \text{ and } [F(S)]_{S\to\infty} = \frac{2\gamma}{\gamma + 1}$$

Experimental investigations of the Nusselt Number and adiabatic temperature by Stalder et al.,[7,8] Dewey,[1] Vrebalovich,[4] and Atassi and Brun[5] indicate that there is excellent agreement with the free molecule theory.

For any given wire, Eq. (15) can be written as

$$(1/k_0)(dR/dI^2R)^{-1}_{\text{corr}} = CPr_0(\rho u/\mu_0)g(S)/S$$

where the constant C is a function of the accommodation coefficient $\bar{\alpha}$, wire length l and wire diameter d. This constant C can be obtained for any particular wire from a freestream calibration of $(1/k_0)(dR/dI^2R)^{-1}_{\text{corr}}$ against $\rho u/\mu_0$, since the function $g(S)/S$ is a constant for $S > 2.5$. In the calibration process, since $M \simeq 6$, Ta/T_0 is taken to approach its asymptotic limit of $2\gamma/(\gamma + 1)$. Hence from Eqs. (5) and (7), ξ tanhl/ξ and ψ_N can be calculated. The calibration yields two quantities; 1) the slope of $(dR/dI^2R)^{-1}_{\text{corr}}$ against ρu or the slope of $(1/k_0)(dR/dI^2R)^{-1}_{\text{corr}}$ against $\rho u/\mu_0$ and 2) the value of $\bar{R}$ for the particular wire. This calibrated value of $\bar{R}$ is expected to take into account the small variations of $(T_{s,\text{vac}} - T_s)$. In the study of wakes behind circular cylinders[3] it was found that $T_s/T_{0,fs}$ is virtually a constant in the wake (though T_s/T_0 exhibits large gradients). Also $T_{s,\text{vac}} = T_{0,fs}$ (unheated flow) and therefore the value of $(T_{s,\text{vac}} - T_s)$ will be approximately the same for freestream and the wake and is of the order 5°C.

Figure 2 shows the calibration of $(dR/dI^2R)^{-1}_{\text{corr}}$ against ρu for two different hot wires. The least squares straight lines through the data points approximately pass through the origin, which indicates excellent agreement with free molecule theory. In Fig. 3 the measured $\bar{R}$ is plotted against ξ and compared with the theoretical prediction for $(T_{s,\text{vac}} - T_s) = 0$. The parameter a is of the order 10^{-1} for most probes. This correction procedure has been used to obtain flow parameters in the near wake of a circular cylinder.[3]

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Optimal Design of Statically Determinate Beams Subject to Displacement and Stress Constraints

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Introduction

As was pointed out by Barnett[1] and stressed by Huang and Tang[2] the optimal design of a beam that is only subject to deflection constraint may not be a well-posed problem. Consider for instance the beam shown in Fig. 1, that carries a concentrated load P and a couple C and which has to be designed for minimum weight, the behavioral constraint being an upper bound on the displacement at the center of the span. Because load and couple have opposite effects on this displacement, it is possible to find a design with vanishing displacement at this point. This property will be conserved when all bending stiffnesses are scaled down by an arbitrary factor; this means that the behavioral constraint may be fulfilled by a design of arbitrarily small structural weight. Of course, this solution of the optimization problem is physically meaningless because the stresses at some cross sections become arbitrarily great. To preclude this possibility, we have

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